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ON LOADING, YIELD AND QUASI-YIELD HYPERSURFACES IN PLASTICITY THEORY

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Abstract—In the context of the author's previously published "simple" theory of plasticity[1] in which no loading or yield surfaces are *assumed* to exist, it is shown that (a) loading surfaces must exist for a plastic material as a result of Caratheodory's theorem on Pfaffian forms, and that (b) a yield hypersurface in state space may be defined as the boundary of the region in which no loading surfaces exist (the elastic region) if this region has a positive volume, otherwise this region degenerates into the quasi-yield hypersurface. The significance of loading and yield (or quasi-yield) hypersurfaces is further explored for one-component loadings, with particular attention to the Bauschinger effect and kinematic hardening.

1. INTRODUCTION

The purpose of this note is to explore further some consequences of a model of elastic-plastic behavior that I proposed in a recent paper[1]. The basis of the model is the notion that plastic deformation takes place only upon loading, not unloading. An isothermal loading rate can be defined either in "strain" space or in "stress" space. For finite deformation the appropriate strain and stress tensors† are, respectively, the Green strain tensor $\mathbf{E}$ and the Piola–Kirchhoff stress tensor $\mathbf{P}$. The isothermal loading rate ϕ in strain space is given by $tr[\mathbf{A}(\mathbf{E}, \theta, \mathbf{q})\dot{\mathbf{E}}]$, and in stress space by $tr[\mathbf{B}(\mathbf{P}, \theta, \mathbf{q})\dot{\mathbf{P}}]$, where $\mathbf{A}(\,)$ and $\mathbf{B}(\,)$ are given tensor functions characterizing the material, θ is the temperature, and $\mathbf{q}$ is a vector of n internal variables governed by rate equations of the form

$$\dot{\mathbf{q}} = \mathbf{r}\langle\phi\rangle. \tag{1}$$

Here $\mathbf{r}$ is an n-vector function of $\mathbf{E}, \theta, \mathbf{q}$ or of $\mathbf{P}, \theta, \mathbf{q}$, depending on the loading space, and $\langle\,\rangle$ is the Macauley bracket.

In the present work the concern is with the relationship between the model described in [1] and theories of plasticity in common use. Consequently, attention will be confined to the "stress-space" version of the model, and, more specifically, to states of small deformation, so that in place of $\mathbf{E}$ and $\mathbf{P}$ the conventional strain and stress tensors $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ will be used (if only for the sake of recognizability). The strain can furthermore be decomposed into elastic and plastic parts:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p, \tag{2}$$

where $\boldsymbol{\epsilon}^e$ is linear in $\boldsymbol{\sigma}$, and the components of $\boldsymbol{\epsilon}^p$ may be used as internal variables. It will be assumed that the only other internal variable is κ, defined by the rate equation

$$\dot{\kappa} = (tr\,\dot{\boldsymbol{\epsilon}}^p\dot{\boldsymbol{\epsilon}}^p)^{1/2}. \tag{3}$$

All processes will be assumed isothermal and dependence on temperature will not be indicated explicitly. The state variables are consequently $\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p, \kappa$; their space will be referred to as *state space*. To formulate the rate equation for $\boldsymbol{\epsilon}^p$ we need tensor functions $\mathbf{M}(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p, \kappa)$ and $\mathbf{B}(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p, \kappa)$ such that

$$\dot{\boldsymbol{\epsilon}}^p = \mathbf{M}\langle tr\,\mathbf{B}\dot{\boldsymbol{\sigma}}\rangle. \tag{4}$$

†The term tensor will be used, unless specified otherwise, to denote a symmetric second-rank tensor.

Equation (3) therefore becomes

$$\dot{\kappa} = (tr\,\mathbf{M}^2)^{1/2}\langle tr\,\mathbf{B}\dot{\boldsymbol{\sigma}}\rangle. \tag{5}$$

Equations (4) and (5) are the basic equations of the model. They incorporate the essential aspect of plastic behavior—the loading-unloading irreversibility—without the usual assumptions as to the existence of loading surfaces, yield surfaces, or stability criteria. However, all these features may be incorporated in the model as follows: (a) Loading hypersurfaces in state space, given by $G(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p, \kappa) = \text{const.}$, exist if there exist functions $\lambda(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p, \kappa)$, $G(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p, \kappa)$ such that

$$\mathbf{B} = \lambda \frac{\partial G}{\partial \boldsymbol{\sigma}}. \tag{6}$$

The projection of a loading hypersurface into the stress subspace (with $\boldsymbol{\epsilon}^p$, κ as parameters) is called a loading surface.

(b) A yield surface exists if **B** or **M** vanishes in a non-vanishing region of state space, called the elastic region. The boundary of the region is the yield hypersurface, given by $F(\boldsymbol{\sigma}, \boldsymbol{\epsilon}^p, \kappa) = 0$, and its projection into the stress subspace (again with $\boldsymbol{\epsilon}^p$, κ as parameters) is a yield surface.

(c) A stability postulate may be expressed in Drucker's form[2]:

$$tr\,\dot{\boldsymbol{\sigma}}\dot{\boldsymbol{\epsilon}}^p \geq 0. \tag{7}$$

As shown in [1], this leads to

$$\mathbf{M} = \mu\mathbf{B},$$

where μ is a positive scalar that may with no loss in generality be taken as unity, since any resulting redefinition of **B** would not alter the basic character of equation (4).

In the next section it will be shown, however, that the existence of loading surfaces can be motivated on compelling physical grounds. In the subsequent section the application of the model to one-component loading will be examined.

2. EXISTENCE OF LOADING AND YIELD SURFACES

Equation (6) is an assertion of the integrability in $\boldsymbol{\sigma}$-space of the Pfaffian differential form $tr\,\mathbf{B}\,\mathrm{d}\boldsymbol{\sigma}$. As is known from Caratheodory's theorem[3], such integrability is in turn equivalent to the existence of points in the neighborhood of a given point $\boldsymbol{\sigma}$ which *cannot* be reached along a path on which $tr\,\mathbf{B}\,\mathrm{d}\boldsymbol{\sigma} = 0$. In the present context, $tr\,\mathbf{B}\,\mathrm{d}\boldsymbol{\sigma} = 0$ means neutral loading. Clearly, unless the stress point is in an elastic region (in which case **B** vanishes), the existence of stress states which may be reached from a given stress state only by loading (with resulting plastic deformation) is in the very essence of plastic behavior. Consequently, we have necessary and sufficient conditions for the existence of a loading hypersurface as defined in the Introduction. Since λ can be absorbed in **M**, equation (4) can be rewritten as

$$\dot{\boldsymbol{\epsilon}}^p = \mathbf{M}\left\langle tr \frac{\partial G}{\partial \boldsymbol{\sigma}} \dot{\boldsymbol{\sigma}} \right\rangle. \tag{8}$$

If Drucker's postulate (7) holds, we have $\mathbf{M} = \mu(\partial G/\partial \boldsymbol{\sigma})$, i.e. normality.

On the other hand, if there exists a non-vanishing region in state space in which the conditions for the existence of a loading hypersurface are not met, i.e. in which $\dot{\boldsymbol{\epsilon}}^p = 0$ for any increment in stress space, then this region is the elastic region, and its boundary is the yield hypersurface.

The arguments of this section are not bound by the small-deformation stress-space restriction, but apply as well to the finite-deformation strain-space and stress-space versions of the model.

3. ONE-COMPONENT LOADINGS

This section deals with loadings describable by one stress component σ; this may be a tensile stress or a shear stress. The conjugate strain is ϵ, the hardening parameter is given as

$$\kappa = \int |d\epsilon^p| \tag{9}$$

and the state variables are σ, ϵ^p, κ. In view of equation (9), only a quarter of the space of these variables, given by

$$-\kappa < \epsilon^p < \kappa \tag{10}$$

represents possible states. The material will be assumed to be stable (i.e. no work-softening), so that the rate equation governing ϵ^p may be written as

$$\dot{\epsilon}^p = \mu \frac{\partial G}{\partial \sigma}\left\langle \frac{\partial G}{\partial \sigma}\dot{\sigma}\right\rangle. \tag{11}$$

Clearly, when plastic deformation occurs, $\mu(\partial G/\partial\sigma)^2$ is the value of the reciprocal plastic modulus $(1/E_p)$† at the given σ, ϵ^p, κ. μ and $\partial G/\partial\sigma$ cannot be separated. Further examination of material behavior will be pursued first for materials having no yield hypersurface, and then for materials having one.

(a) *No yield surface.* If no yield hypersurface exists, then, by definition, there is no region of non-vanishing volume in which $(\partial G/\partial\sigma)(\sigma, \epsilon^p, \kappa) = 0$. Since, however loading can proceed in both the positive and negative directions, $(\partial G/\partial\sigma)$ must take on both positive and negative values. Consequently, there is a surface on which $(\partial G/\partial\sigma) = 0$; let it be denoted by $\sigma = g(\epsilon^p, \kappa)$, and called the *quasi-yield surface.* Consider monotonic loading in the positive direction; we have $\epsilon^p = \kappa$, and the projection of the quasi-yield surface in the $\sigma - \epsilon^p$ plane is the curve $\sigma = g(\epsilon^p, \epsilon^p) \equiv g'(\epsilon^p)$. Clearly when $\sigma > g'(\epsilon^p)$, plastic deformation will occur only when $\dot{\sigma} > 0$, and vice versa. The resulting loading-unloading diagram is shown in Fig. 1.

If unloading proceeds below B, reverse plastic deformation will result—a very prominent form of the Bauschinger effect. If unloading stops at some point B' above B, plastic deformation will occur immediately upon reloading. This kind of behavior was observed in Phillips[4] in graphite.

Note that the *shape* of the loading curves depends on the values of $\mu(\partial G/\partial\sigma)^2$ away from the quasi-yield surface. Let us define

$$f(\sigma, \epsilon^p, \kappa) \equiv \mu\left(\frac{\partial G}{\partial \sigma}\right)^2.$$

Then

$$\frac{d\epsilon^p}{d\sigma} = f(\sigma, \epsilon^p, \kappa), \quad [\sigma - g(\epsilon^p, \kappa)]\dot{\sigma} > 0$$
$$= 0, \quad [\sigma - g(\epsilon^p, \kappa)]\dot{\sigma} < 0. \tag{12}$$

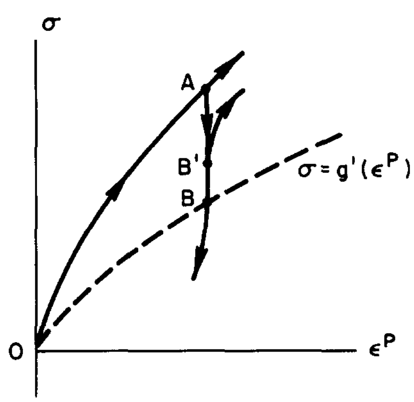

Fig. 1.

†$(1/E_p) = (1/E_t) - (1/E)$, where E is the elastic modulus and E_t is the tangent modulus of the loading curve in the $\sigma - \epsilon$ plane.

(b) *Yield surface exists.* Let the elastic region be denoted by $g_-(\epsilon^P, \kappa) < \sigma < g_+(\epsilon^P, \kappa)$. As before, let $g'_\pm(\epsilon^P) \equiv g_\pm(\epsilon^P, \epsilon^P)$. Then $\sigma = g'_\pm(\epsilon^P)$ represents the upper and lower yield curves in monotonic loading. Their significance can be seen in a diagram similar to that of the previous section, shown in Fig. 2.

If unloading extends no lower than B, plastic deformation occurs immediately upon reloading. If it stops between B and C and reloading takes place, plastic deformation occurs upon return to B. If unloading proceeds below C, the Bauschinger effect takes place.

By definition $\sigma_y = g'_+(0)$. By assumed symmetry, $g'_-(0) = -\sigma_y$. By further symmetry, if we consider monotonic loading in the negative direction, so that $\epsilon^P = -\kappa$, we should have $g_+(-\kappa, \kappa) = -g'_-(\kappa)$ and $g_-(-\kappa, \kappa) = -g'_+(\kappa)$. More generally still, it can be argued that

$$g_-(\epsilon^P, \kappa) = -g_+(-\epsilon^P, \kappa).$$

Let us now consider the significance of $g_+(0, \kappa)$, to be denoted by $g_0(\kappa)$. We may vary κ by finite amounts while keeping ϵ^P at zero by performing cyclic loading as shown in Fig. 3.

The points $A_0, A_1, A_2, \ldots$, correspond to $\epsilon^P = 0$ and $\kappa = 0, 2\Delta, 4\Delta, \ldots$. We may plot the yield stresses as a function of κ, as in Fig. 4.

The curves resemble fatigue curves, with hardening rather than number of cycles as abscissa.

Let us now consider the consequences of the hypothesis that the yield hypersurface obeys a fairly general kinematic hardening rule of the form

$$\bar{F}(\boldsymbol{\sigma} - \mathbf{h}(\boldsymbol{\epsilon}^P, \kappa), \kappa) = 0, \tag{13}$$

where $\mathbf{h}(\,)$ is a tensor function.

For the transformation $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$, $\epsilon^P \to -\epsilon^P$, $\kappa \to \kappa$ to leave $\bar{F}$ unchanged, $\mathbf{h}$ must be an odd function of $\boldsymbol{\epsilon}^P$, and $\bar{F}$ an even function of $\boldsymbol{\sigma} - \mathbf{h}$. In one-component loading equation (13) reduces to

$$\bar{F}(\sigma - h(\epsilon^P, \kappa), \kappa) = 0. \tag{14}$$

It can be solved in the form

$$|\sigma - h(\epsilon^P, \kappa)| = g_0(\kappa),$$

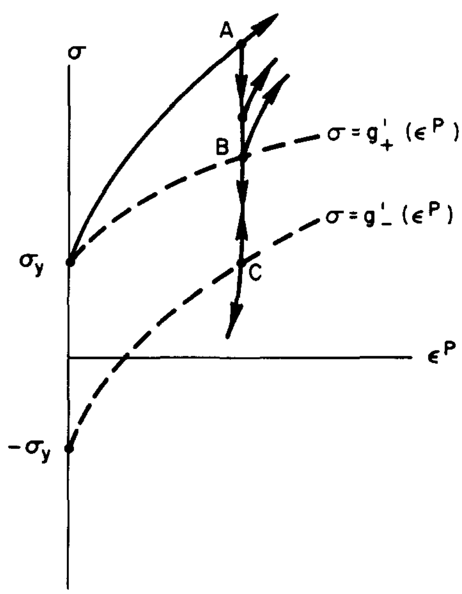

Fig. 2.

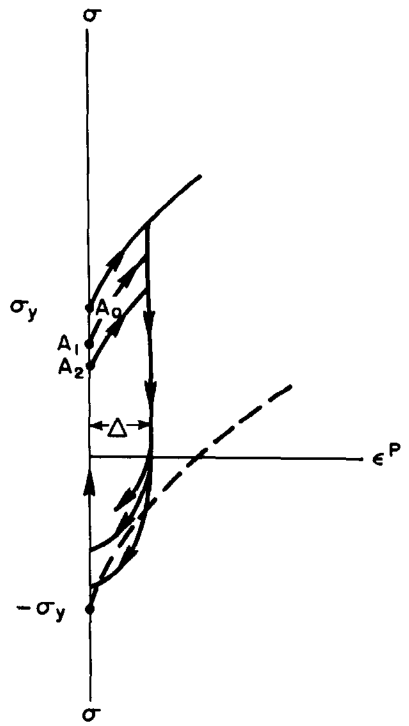

Fig. 3.

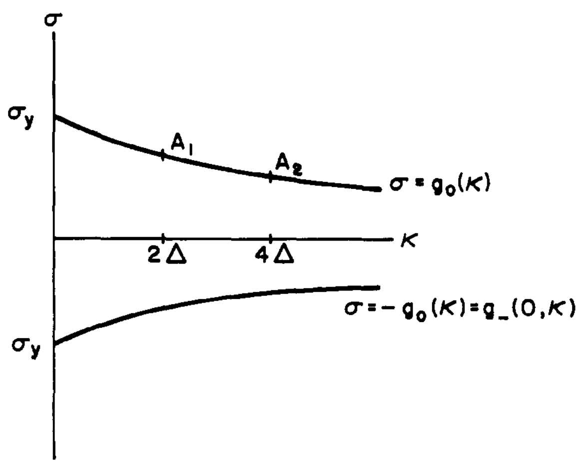

Fig. 4.

whence we obtain

$$g_{\pm}(\epsilon^P, \kappa) = \pm g_0(\kappa) + h(\epsilon^P, \kappa). \tag{15}$$

Clearly the odd dependence of h on ϵ^P is necessary and sufficient for the symmetry conditions discussed above.

An immediate consequence of equation (15) is

$$g'_-(\epsilon^P) = g'_+(\epsilon^P) - 2g_0(\epsilon^P). \tag{16}$$

In other words, from determinations of the upper yield curves for monotonic loading and for cyclic loading, the lower yield curve for monotonic loading (i.e. the onset of the Bauschinger effect) can be predicted for material obeying kinematic hardening of the form (13). This result provides a simple experimental test of the hypothesis of kinematic hardening.

Furthermore, if we set $g_0(\kappa) = 0$, the elastic region degenerates into a surface, namely, the quasi-yield surface discussed in subsection (a). Equation (13) consequently represents a quasi-yield surface if $\bar{F}$ depends on the first argument only and vanishes when the argument vanishes, i.e.

$$\bar{F}(\boldsymbol{\sigma} - \mathbf{h}(\boldsymbol{\epsilon}^p, \kappa)) = 0; \quad \bar{F}(0) = 0.$$

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